Q.1

Let $r = \sqrt{x^2+y^2+z^2}$ & $\vec{r} = (x, y, z)$.

Define the vector field $F : \mathbb{R}^3 \setminus \{\vec{0}\} \to \mathbb{R}^3$ by

$$F(x,y,z) = -\frac{1}{4\pi} \nabla\left(\frac{1}{r}\right)$$

(a) Show that $F(x,y,z) = \frac{1}{4\pi} \frac{\vec{r}}{r^3}$

(b) Show that $\operatorname{div}(F)(x,y,z) = 0$ for $(x,y,z) \neq (0,0,0)$.

Let $V \subseteq \mathbb{R}^3$ be open & bounded. Suppose $(0,0,0) \in V$.

(c) Show that $\int_{\partial V} F \cdot \vec{n} \, d\sigma = 1$.

Solution:

(a) $\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \frac{\partial}{\partial y}\left(\frac{1}{r}\right), \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right)$

$= \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\right), \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\left(\frac{1}{r}\right), \frac{\partial r}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)$

$= \left(\frac{x}{r}\left(-\frac{1}{r^2}\right), \frac{y}{r}\left(-\frac{1}{r^2}\right), \frac{z}{r}\left(-\frac{1}{r^2}\right)\right)$

$= -\frac{1}{r^3}(x, y, z)$

$= -\frac{\vec{r}}{r^3}$

$\therefore \vec{F}(x, y, z) = \frac{1}{4\pi}\frac{\vec{r}}{r^3}$

(b) $\operatorname{div}(F) = \frac{1}{4\pi}\left(\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right)$

$= \frac{1}{4\pi}\frac{1}{r^6}\left(r^3 - x(3r^2)\left(\frac{x}{r}\right) + r^3 - y(3r^2)\left(\frac{y}{r}\right) + r^3 - z(3r^2)\left(\frac{z}{r}\right)\right)$

$= \frac{1}{4\pi}\frac{1}{r^6}\left(3r^3 - 3r(x^2+y^2+z^2)\right)$

$= \frac{1}{4\pi}\frac{1}{r^6}(3r^3 - 3r^3)$

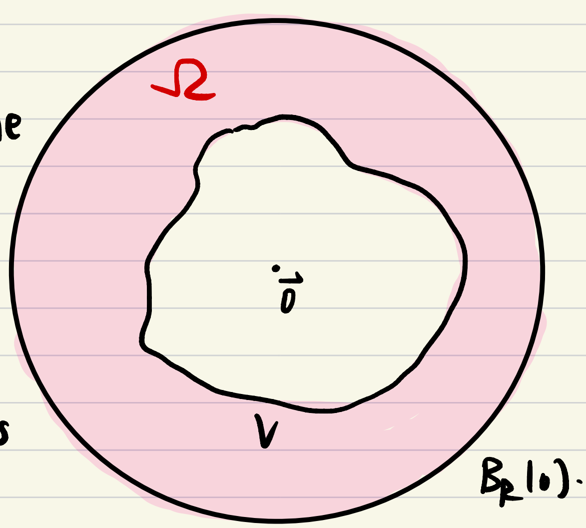
$= 0$

(c) Let $B_R(0)$ be an open ball s.t. $\overline{V} \subseteq B_R(0)$.

Then $\Omega = B_R(0) \setminus \overline{V}$ is an open region that doesn't contain the origin. By the divergence theorem,

$$\int_{\partial\Omega} F \cdot \vec{n}\, d\sigma = \int_{\Omega} \operatorname{div}(F)\, dV = 0$$

since $\operatorname{div}(F) = 0$. Note that $\vec{n}$ denotes the outward unit normal of Ω.



However, $\vec{n}\big|_{\partial V}$ is an inward unit normal of ∂V.

$$\therefore \quad 0 = \int_{\partial\Omega} F \cdot d\vec{n}\, d\sigma$$

$$= \int_{\partial B_R(0)} F \cdot \vec{n}\, d\sigma - \int_{\partial V} F \cdot \vec{n}\, d\sigma$$

$$\int_{\partial V} F \cdot \vec{n}\, d\sigma = \int_{\partial B_R(0)} F \cdot \vec{n}\, d\sigma$$

$$= \int_{\partial B_R(0)} \frac{1}{4\pi} \frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r}\, d\sigma$$

$$= \int_{\partial B_R(0)} \frac{1}{4\pi} \frac{1}{r^2}\, d\sigma \qquad (\vec{r} \cdot \vec{r} = r^2)$$

$$= \frac{1}{4\pi} \frac{1}{R^2} \int_{\partial B_R(0)} 1\, d\sigma \qquad (r \equiv R \text{ on } \partial B_R(0))$$

$$= \frac{1}{4\pi} \frac{1}{R^2} 4\pi R^2$$

$$= 1.$$

Q.2

Let $U \subseteq \mathbb{R}^3$ be an open set. Let $G: U \to \mathbb{R}^3$ be a continuous vector field. Consider the following equation:

$$\operatorname{curl} F = G \qquad (*)$$

where $F: U \to \mathbb{R}^3$ is a C^1 vector field.

Assume that U is simply connected.

(a) Show that if $(*)$ has a solution F, then $\operatorname{div}(G) = 0$.

(b) Show that if $F_1, F_2 : U \to \mathbb{R}^2$ solve $(*)$, then

$$F_2 = F_1 + \nabla f$$

for some C^2 function $f: U \to \mathbb{R}^3$.

Idea of getting the solution to $(*)$:

Suppose $\operatorname{div} G = 0$. If we have one solution H to $(*)$, then $H + \nabla f$ is also a solution of $(*)$. We can always find a C^2 function f s.t. $F = H + \nabla f$ satisfies $F_3 = 0$, i.e. $\frac{\partial f}{\partial z} = -H_3$ (By FTC). So we may assume $F = (F_1, F_2, 0)$.

So $(*)$ becomes

$$\begin{cases} -\dfrac{\partial F_2}{\partial z} = G_1 \\ \dfrac{\partial F_1}{\partial z} = G_2 \\ \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = G_3 . \end{cases}$$

So the problem becomes solving the previous system of PDE.

(c) Try the construction on $G(x,y,z) = (yz, xz, xy)$.

Solution:

(a) If $G = \mathrm{curl}(F)$, then $\mathrm{div}(G) = \mathrm{div}(\mathrm{curl}(F)) = 0$.

(b) If F_1, F_2 solves $(*)$, then

$$\begin{cases} \mathrm{curl}\, F_1 = G & — ① \\ \mathrm{curl}\, F_2 = G & — ② \end{cases}$$

② − ①: $\mathrm{curl}(F_2 - F_1) = 0$.

This implies that that $F_2 - F_1$ satisfies the compatibility conditions. Since $\mathcal{U}$ is simply connected & $F_2 - F_1$ is C^1, we know that $F_2 - F_1$ is conservative, i.e. $\exists$ C^1 function $f : \mathcal{U} \to \mathbb{R}$ s.t. $F_2 - F_1 = \nabla f$.

This equality also implies that all partial derivatives are C^1, so f is C^2.

(c)

$$\begin{cases} -\dfrac{\partial F_2}{\partial z} = yz & — ① \\ \dfrac{\partial F_1}{\partial z} = xz & — ② \\ \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F}{\partial y} = xy & — ③ \end{cases}$$

By ①, $F_2 = -\frac{1}{2} y z^2 + C_1(x, y)$

By ②, $F_1 = -\frac{1}{2} x z^2 + C_2(x, y)$

By ③, $\dfrac{\partial C_1}{\partial x} - \dfrac{\partial C_2}{\partial y} = xy$

There is some freedom in choosing C_1, C_2. For example, choose

$C_2 \equiv 0$ & $C_1 = \frac{1}{2} y x^2$

$\therefore F(x,y,z) = \left(-\frac{1}{2} y z^2 + \frac{1}{2} y x^2, -\frac{1}{2} x z^2, 0\right)$ is one solution.

Q.3

Let E be a plane in $\mathbb{R}^3$ with a unit normal vector $\hat{n}$. Fix $a \in E$. For each $r > 0$, define $A(r) = \{x \in E \mid |x - a| \leq r\}$. Suppose that the boundary $\partial A(r)$ carries the orientation induced by $\hat{n}$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 vector field. Show that

$$\operatorname{curl} F(a) \cdot n = \lim_{r \to 0^+} \frac{1}{\operatorname{Area}(A(r))} \int_{\partial A(r)} F \cdot d\vec{r}$$

Solution:

$$\left| \frac{1}{\text{Area}(A(r))} \int_{\partial A(r)} F \cdot d\vec{r} - \text{curl}\, F(a) \cdot n \right|$$

$$= \left| \frac{1}{\text{Area}(A(r))} \int_{A(r)} \text{curl}\, F \cdot n \, d\sigma - \text{curl}\, F(a) \cdot n \right| \quad \text{(Stokes' thm)}$$

$$= \left| \frac{1}{\text{Area}(A(r))} \left(\int_{A(r)} (\text{curl}\, F \cdot n - \text{curl}\, F(a) \cdot n) \, d\sigma \right) \right|$$

$$\leq \frac{1}{\text{Area}(A(r))} \int_{A(r)} \left| \text{curl}\, F \cdot n - \text{curl}\, F(a) \cdot n \right| d\sigma \quad - (*)$$

Since $\text{curl}(F)$ & $\hat{n}$ are continuous, $\forall \varepsilon > 0, \exists \delta$ s.t. when $|x - a| < \delta$, then $\left| \text{curl}\, F \cdot n - \text{curl}\, F(a) \cdot n \right| < \varepsilon$.

Therefore, if $0 < r < \delta$, then $\forall x \in A(r)$,

$$\left| \text{curl}\, F(x) \cdot n - \text{curl}\, F(a) \cdot n \right| < \varepsilon$$

Continuing the estimate above

$$(*) \leq \frac{1}{\text{Area}(A(r))} \int_{A(r)} \varepsilon \, d\sigma = \frac{1}{\text{Area}(A(r))} \cdot \text{Area}(A(r))\, \varepsilon = \varepsilon$$

Then the result follows by the def. of limit.